RECOGNISING ACHIEVEMENT

ADVANCED GCE

MATHEMATICS

Mechanics 4

4731

Candidates answer on the answer booklet.

OCR supplied materials:
- 8 page answer booklet (sent with general stationery)
- List of Formulae (MF1)

Other materials required:
- Scientific or graphical calculator

Thursday 23 June 2011
Morning

Duration: 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet. Please write clearly and in capital letters.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Answer **all** the questions.
- Do **not** write in the bar codes.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.
- The acceleration due to gravity is denoted by g m s^{-2}. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.
- You are permitted to use a scientific or graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- **You are reminded of the need for clear presentation in your answers.**
- The total number of marks for this paper is **72**.
- This document consists of **4** pages. Any blank pages are indicated.

1 When the power is turned off, a fan disk inside a jet engine slows down with constant angular deceleration $0.8\,\text{rad}\,\text{s}^{-2}$.

(i) Find the time taken for the angular speed to decrease from $950\,\text{rad}\,\text{s}^{-1}$ to $750\,\text{rad}\,\text{s}^{-1}$. **[2]**

(ii) Find the angle through which the disk turns as the angular speed decreases from $220\,\text{rad}\,\text{s}^{-1}$ to $200\,\text{rad}\,\text{s}^{-1}$. **[2]**

(iii) Find the time taken for the disk to make the final 10 revolutions before coming to rest. **[3]**

2 A straight rod AB has length a. The rod has variable density, and at a distance x from A its mass per unit length is $k\mathrm{e}^{-\frac{x}{a}}$, where k is a constant. Find, in an exact form, the distance of the centre of mass of the rod from A. **[7]**

3 A uniform rod XY, of mass 5 kg and length 1.8 m, is free to rotate in a vertical plane about a fixed horizontal axis through X. The rod is at rest with Y vertically below X when a couple of constant moment is applied to the rod. It then rotates, and comes instantaneously to rest when XY is horizontal.

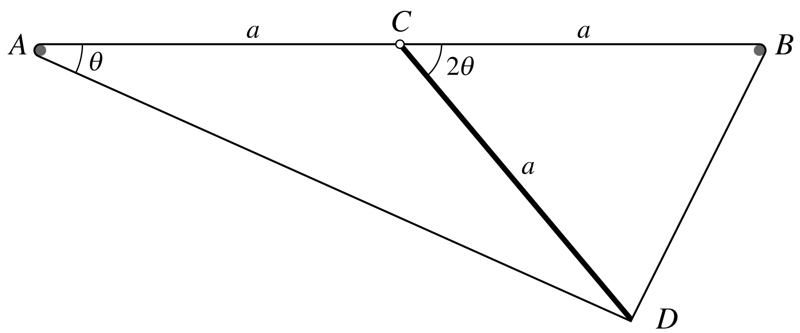
(i) Find the moment of the couple. **[4]**

(ii) Find the angular acceleration of the rod

(a) immediately after the couple is first applied, **[3]**

(b) when XY is horizontal. **[2]**

4



Two small smooth pegs A and B are fixed at a distance $2a$ apart on the same horizontal level, and C is the mid-point of AB. A uniform rod CD, of mass m and length a, is freely pivoted at C and can rotate in the vertical plane containing AB, with D below the level of AB. A light elastic string, of natural length a and modulus of elasticity $3mg$, passes round the peg A and its ends are attached to C and D. Another light elastic string, of natural length a and modulus of elasticity $4mg$, passes round the peg B and its ends are also attached to C and D. The angle CAD is θ, where $0 < \theta < \frac{1}{2}\pi$, so that the angle BCD is 2θ (see diagram).

(i) Taking AB as the reference level for gravitational potential energy, show that the total potential energy of the system is

$$\tfrac{1}{2}mga(14 - 2\cos 2\theta - \sin 2\theta).$$ **[5]**

(ii) Find the value of θ for which the system is in equilibrium. **[3]**

(iii) Determine whether this position of equilibrium is stable or unstable. **[2]**

5 The region inside the circle $x^2 + y^2 = a^2$ is rotated about the x-axis to form a uniform solid sphere of radius a and volume $\frac{4}{3}\pi a^3$. The mass of the sphere is $10M$.

(i) Show by integration that the moment of inertia of the sphere about the x-axis is $4Ma^2$. (You may assume the standard formula $\frac{1}{2}mr^2$ for the moment of inertia of a uniform disc about its axis.) **[6]**

The sphere is free to rotate about a fixed horizontal axis which is a diameter of the sphere. A particle of mass M is attached to the lowest point of the sphere. The sphere with the particle attached then makes small oscillations as a compound pendulum.

(ii) Find, in terms of a and g, the approximate period of these oscillations. **[5]**

6 Two ships P and Q are moving on straight courses with constant speeds. At one instant Q is 80 km from P on a bearing of 220°. Three hours later, Q is 36 km due south of P.

(i) Show that the velocity of Q relative to P is 19.1 km h^{-1} in the direction with bearing 063.8° (both correct to 3 significant figures). **[5]**

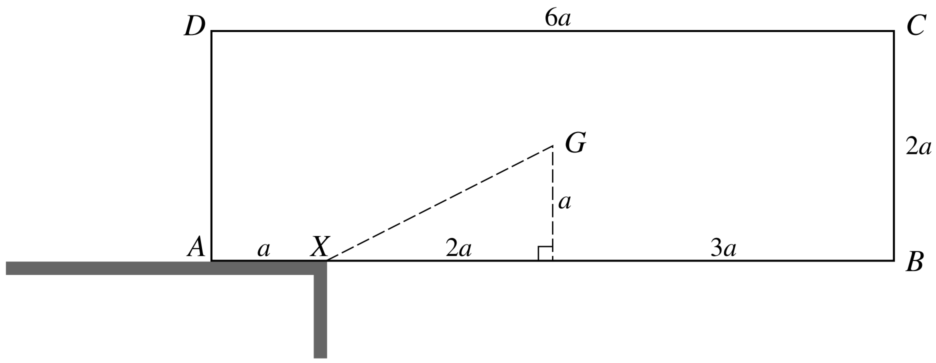
(ii) Find the shortest distance between the two ships in the subsequent motion. **[2]**

Given that the speed of P is 28 km h^{-1} and Q is travelling in the direction with bearing 105°, find

(iii) the bearing of the direction in which P is travelling, **[3]**

(iv) the speed of Q. **[2]**

7



A uniform rectangular block of mass m and cross-section $ABCD$ has $AB = CD = 6a$ and $AD = BC = 2a$. The point X is on AB such that $AX = a$ and G is the centre of $ABCD$. The block is placed with AB perpendicular to the straight edge of a rough horizontal table. AX is in contact with the table and XB overhangs the edge (see diagram). The block is released from rest in this position, and it rotates without slipping about a horizontal axis through X.

(i) Find the moment of inertia of the block about the axis of rotation. **[3]**

For the instant when XG is horizontal,

(ii) show that the angular acceleration of the block is $\dfrac{3\sqrt{5}g}{25a}$, **[2]**

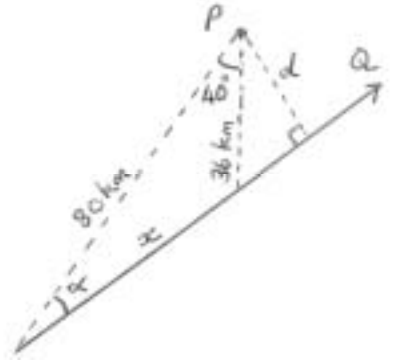
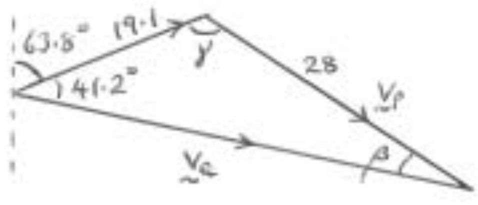
(iii) find the angular speed of the block, **[3]**

(iv) show that the force exerted by the table on the block has magnitude $\dfrac{2\sqrt{70}}{25}mg$. **[8]**

1 **(i)**	Using $\omega_2 = \omega_1 + \alpha t$, $750 = 950 - 0.8t$ Time taken is 250 s	M1 A1 [2]	
(ii)	Using $\omega_2^2 = \omega_1^2 + 2\alpha\theta$, $200^2 = 220^2 - 1.6\theta$ Angle is 5250 rad	M1 A1 [2]	
(iii)	Angle is 20π rad Using $\theta = \omega_2 t - \frac{1}{2}\alpha t^2$, $20\pi = 0 + 0.4t^2$ Time taken is 12.5 s (3 sf)	B1 M1 A1 [3]	or equivalent; e.g. finding $\omega_1 = 10.03$ and then $t = \omega_1 \div 0.8$ *Accept* $\sqrt{50\pi}$ or $5\sqrt{2\pi}$
2	$m = \int_0^a k\,\mathrm{e}^{-\frac{x}{a}}\,\mathrm{d}x$ $= k\left[-a\,\mathrm{e}^{-\frac{x}{a}}\right]_0^a \quad \left(= ka(1-\mathrm{e}^{-1})\right)$ $m\bar{x} = \int_0^a xk\,\mathrm{e}^{-\frac{x}{a}}\,\mathrm{d}x$ $= k\left[-ax\,\mathrm{e}^{-\frac{x}{a}} - a^2\mathrm{e}^{-\frac{x}{a}}\right]_0^a$ $= ka^2(1-2\mathrm{e}^{-1})$ $\bar{x} = \frac{ka^2(1-2\mathrm{e}^{-1})}{ka(1-\mathrm{e}^{-1})}$ $= \frac{a(1-2\mathrm{e}^{-1})}{1-\mathrm{e}^{-1}} = \frac{a(\mathrm{e}-2)}{\mathrm{e}-1}$	M1 A1 M1 M1 A1 A1 A1 [7]	For $\int \mathrm{e}^{-\frac{x}{a}}\,\mathrm{d}x$ For $-a\,\mathrm{e}^{-\frac{x}{a}}$ For $\int x\,\mathrm{e}^{-\frac{x}{a}}\,\mathrm{d}x$ Integration by parts For $-ax\,\mathrm{e}^{-\frac{x}{a}} - a^2\mathrm{e}^{-\frac{x}{a}}$ For $a^2(1-2\mathrm{e}^{-1})$ or exact equivalent
3 **(i)**	WD by couple is $C \times \frac{\pi}{2}$ Change in PE is $5 \times 9.8 \times 0.9$ By conservation of energy, $C \times \frac{\pi}{2} = 5 \times 9.8 \times 0.9$ Moment of couple is 28.1 Nm (3 sf)	B1 B1 M1 A1 [4]	 Must clearly be PE (not moment) Equation involving WD and PE
(ii) **(a)**	$I = \frac{4}{3} \times 5 \times 0.9^2 \quad (= 5.4)$ $28.075 = 5.4\alpha$ Angular acceleration is $5.20\ \mathrm{rad\,s^{-2}}$ (3 sf)	B1 M1 A1 ft [3]	*Can be earned anywhere in the question* Applying $C = I\alpha$ ft is $C \div I$
(ii) **(b)**	$28.075 - 5 \times 9.8 \times 0.9 = 5.4\alpha$ Angular acceleration is $(-)\ 2.97\ \mathrm{rad\,s^{-2}}$ (3 sf)	M1 A1 [2]	Rotational equation of motion (3 terms) (*Allow 1.8 instead of 0.9 etc*)

4 **(i)**	GPE is $-mg(\frac{1}{2}a\sin 2\theta)$	B1	*Negative sign is essential, but may be implied later*
	EPE is $\frac{3mg}{2a}AD^2 + \frac{4mg}{2a}BD^2$	M1	
	$= \frac{3mg}{2a}(2a\cos\theta)^2 + \frac{4mg}{2a}(2a\sin\theta)^2$	A1	Any correct form
	$= mga(6\cos^2\theta + 8\sin^2\theta)$		
	$= mga(3 + 3\cos 2\theta + 4 - 4\cos 2\theta)$	M1	Expressing EPE in terms of $\cos 2\theta$
	$= mga(7 - \cos 2\theta)$		
	Total PE is $V = mga(7 - \cos 2\theta) - \frac{1}{2}mga\sin 2\theta$		
	$= \frac{1}{2}mga(14 - 2\cos 2\theta - \sin 2\theta)$	A1 ag [**5**]	
(ii)	$\frac{dV}{d\theta} = \frac{1}{2}mga(4\sin 2\theta - 2\cos 2\theta)$	B1	
	$\frac{dV}{d\theta} = 0$ when $4\sin 2\theta = 2\cos 2\theta$		
	$\tan 2\theta = 0.5$	M1	Equating to zero and solving
	$\theta = 0.232$ (3 sf)	A1 [**3**]	Accept 13.3°
(iii)	$\frac{d^2V}{d\theta^2} = \frac{1}{2}mga(8\cos 2\theta + 4\sin 2\theta)$		
	When $\theta = 0.232$, $\frac{d^2V}{d\theta^2} > 0$	M1	
	So the equilibrium is stable	A1 [**2**]	Fully correct working only

5 (i)	$(\frac{4}{3}\pi a^3)\rho = 10M$, so $\rho = \dfrac{15M}{2\pi a^3}$	M1	
	$I = \sum \frac{1}{2}(\rho\pi y^2 \delta x)y^2 = \frac{1}{2}\rho\pi \int y^4\, dx$	M1	For $\int y^4\, dx$
	$= \frac{1}{2}\rho\pi \int_{-a}^{a} (a^2 - x^2)^2\, dx$	A1	Correct integral expression including limits
	$= \frac{1}{2}\rho\pi \left[a^4 x - \frac{2}{3}a^2 x^3 + \frac{1}{5}x^5 \right]_{-a}^{a}$	A1	For $a^4 x - \frac{2}{3}a^2 x^3 + \frac{1}{5}x^5$
	$= \frac{1}{2}\rho\pi \left(a^5 - \frac{2}{3}a^5 + \frac{1}{5}a^5 \right) \times 2$		
	$= \dfrac{8}{15}\rho\pi a^5$	A1	
	$= \dfrac{8}{15} \times \dfrac{15M}{2\pi a^3} \times \pi a^5 = 4Ma^2$	A1 ag [**6**]	
(ii)	MI is $4Ma^2 + Ma^2$	M1	
	$= 5Ma^2$	A1	
	$-Mga\sin\theta = 5Ma^2\ddot{\theta}$	M1	Equation of motion
	$\ddot{\theta} \approx -\dfrac{g}{5a}\theta$		
	Period is $2\pi\sqrt{\dfrac{5a}{g}}$	M1 A1 [**5**]	Obtaining period
	Alternative for last 3 marks of (ii)		
	$11M\bar{x} = 10M(0) + Ma$ — M1		Finding centre of mass
	$\bar{x} = \frac{1}{11}a$		
	Period is $2\pi\sqrt{\dfrac{I}{mgh}} = 2\pi\sqrt{\dfrac{5Ma^2}{11Mg\frac{1}{11}a}}$ — M1		Using formula *Dependent on previous M1*
	$= 2\pi\sqrt{\dfrac{5a}{g}}$ — A1		*Note* $2\pi\sqrt{\dfrac{I}{Mgh}} = 2\pi\sqrt{\dfrac{5Ma^2}{Mga}}$ *is M0*

6 (i)	As viewed from *P* 	M1	Suitable diagram showing relative velocity *May be implied*
	$x^2 = 80^2 + 36^2 - 2\times 80\times 36\cos 40°$ $x = 57.30$	M1	
	Relative velocity has magnitude $\frac{x}{3} = 19.1 \text{ km h}^{-1}$	A1 ag	
	$\frac{\sin\alpha}{36} = \frac{\sin 40°}{57.30}$ $\alpha = 23.82°$	M1	Or other valid method for finding a relevant angle
	Relative velocity has bearing $40 + \alpha = 063.8°$	A1 ag [**5**]	
	OR, using components, Diagram M1 East $\frac{80\sin 40°}{3}$ $(= 17.14)$ M1 North $\frac{80\cos 40° - 36}{3}$ $(= 8.428)$ M1 Speed $\sqrt{17.14^2 + 8.428^2} = 19.1$ A1 ag Bearing $\tan^{-1}\frac{17.14}{8.428} = 063.8°$ A1 ag		Implied by both components correct
(ii)	Shortest distance $d = 80\sin 23.82°$ $= 32.3$ km (3 sf)	M1 A1 [**2**]	or $36\sin 63.8°$
(iii)		M1	Velocity diagram *May be implied* (28 opposite a known angle between sides with positive and negative slopes)
	$\frac{\sin\beta}{19.10} = \frac{\sin 41.18°}{28}$ $\beta = 26.69°$	M1	
	Bearing of *P* is $105 + \beta = 131.7°$ (1 dp)	A1 [**3**]	*Using components for (iii) and (iv)* *M2A1 for* $\theta = 131.7°$ *or* $v = 39.4$ *M1A1 for other quantity*
(iv)	$\frac{v_Q}{\sin 112.13°} = \frac{28}{\sin 41.18°}$	M1	Or other valid method for finding speed
	Speed of *Q* is 39.4 km h^{-1} (3 sf)	A1 [**2**]	

7 (i)	$XG = \sqrt{5}a$	B1	For $I_G = \frac{1}{3}m\{a^2 + (3a)^2\}$
	$I = \frac{1}{3}m\{a^2 + (3a)^2\} + m(\sqrt{5}a)^2$	M1	Using parallel axes rule
	$= \frac{25}{3}ma^2$	A1 [3]	
	OR, other complete method, e.g. M1 $\frac{4}{3}(\frac{1}{6}m)\left((\frac{1}{2}a)^2 + a^2\right) + \frac{4}{3}(\frac{5}{6}m)\left((\frac{5}{2}a)^2 + a^2\right)$ A1 $I = \frac{25}{3}ma^2$ A1		Correct expression for I
(ii)	$mg(\sqrt{5}a) = I\alpha$ $\sqrt{5}mga = \frac{25}{3}ma^2\alpha$	M1	*Allow, e.g.* $mg(2a) = I\alpha$
	$\alpha = \frac{3\sqrt{5}g}{25a}$	A1 ag [2]	
(iii)	$\frac{1}{2}I\omega^2 = mga$	M1	Equation involving KE and PE
	$\frac{25}{6}ma^2\omega^2 = mga$	A1 ft	
	$\omega = \sqrt{\frac{6g}{25a}}$	A1 [3]	
(iv)	$H = m(XG)\omega^2$	M1 A1	For using acceleration $r\omega^2$ Or (F parallel to BA, θ is angle GXB) $F - mg\sin\theta = m\left((AG)\omega^2\cos\theta - (AG)\alpha\sin\theta\right)$
	$= m(\sqrt{5}a)\left(\frac{6g}{25a}\right)$ $= \frac{6\sqrt{5}}{25}mg$	A1 ft	ft from incorrect ω only Or $F = \frac{mg(2\sqrt{5}+12)}{25}$
	$mg - V = m(XG)\alpha$ $V = mg - m(\sqrt{5}a)\left(\frac{3\sqrt{5}g}{25a}\right)$	M1 A1	For using acceleration $r\alpha$ Or (R parallel to AD) $mg\cos\theta - R = m\left((AG)\omega^2\sin\theta + (AG)\alpha\cos\theta\right)$
	$= \frac{2}{5}mg$	A1	Or $R = \frac{mg(4\sqrt{5}-6)}{25}$
	Force has magnitude $\sqrt{H^2 + V^2}$		Or $\sqrt{F^2 + R^2}$
	$= \frac{2}{25}mg\sqrt{(3\sqrt{5})^2 + 5^2}$	M1	
	$= \frac{2\sqrt{70}}{25}mg$	A1 ag [8]	